VOLUME 81 SEPARATE No. 715

PROCEEDINGS

AMERICAN SOCIETY
OF
CIVIL ENGINEERS

JUNE, 1955

AMERICAN SOCIETY OF CIVIL ENGINEERS FOUNDED 1852

COMPUTATION OF TIDES AND TIDAL CURRENTS-UNITED STATES PRACTICE

by H. A. Einstein, M. ASCE,
and R. A. Fuchs

HYDRAULICS DIVISION

{Discussion open until October 1, 1955}

Headquarters of the Society
33 W. 39th St.
New York 18, N. Y.

PRICE $0.50 PER COPY

THIS PAPER

--represents an effort by the Society to deliver technical data direct from the author to the reader with the greatest possible speed. To this end, it has had none of the usual editing required in more formal publication procedures.

Readers are invited to submit discussion applying to current papers. For this paper the final date on which a discussion should reach the Manager of Technical Publications appears on the front cover.

Those who are planning papers or discussions for "Proceedings" will expedite Division and Committee action measurably by first studying "Publication Procedure for Technical Papers" (Proceedings – Separate No. 290). For free copies of this Separate—describing style, content, and format—address the Manager, Technical Publications, ASCE.

This paper was published at 1745 S. State Street, Ann Arbor, Mich., by the American Society of Civil Engineers. Editorial and General Offices are at 33 West Thirty-ninth Street, New York 18, N. Y.

COMPUTATION OF TIDES AND TIDAL CURRENTS—UNITED STATES PRACTICE

H.A. Einstein,* M. ASCE, and R.A. Fuchs**

The authors of this paper have recently made a survey of past and present calculation methods used for the prediction of tidal stages and flows in canals and estuaries. The results of this study conducted under contract for the Committee on Tidal Hydraulics, Corps of Engineers, U.S. Army, are in part offered in this paper.

As the title says, this report is limited to the U.S. Practice, as developed and applied before the second World War. Very little work was done in this field during and immediately after the war until the Committee on Tidal Hydraulics revived the interest, a result of this effort being the present symposium. As no new calculation methods were developed and tested which could be termed original during this study, the present paper will be restricted to the description and discussion of the prominent older methods, where there will be some occasion to indicate where the methods may be modified for added accuracy or simplicity.

Before these methods are discussed in detail, it appears to be necessary to introduce some of the concepts and the physical background of all such calculations. The general problem of a tidal flow with or without a constant flow component is mathematically an extremely complex problem and can not be solved rigorously in most cases. Most solutions depend on the introduction of various simplifying assumptions which may or may not be permissible in any particular application. A short resume may be given first of all the complicating aspects of the tidal flow problem in general. The calculation methods referred to later will be shown to describe only partially the complicated flows.

The General Tidal Flow and the Tidal Basin

Most people associate with the term "tidal flow" reversing flow caused by tidal action. It can easily be seen that any such flow is always affected by other influences, too, such as friction or the inflow of a river. It is much more appropriate, therefore, to include into our problem all flows which are to some degree affected by tidal action, which is the wave condition generated in large bodies of water by the variation of the attraction by the moon and the sun as these celestial bodies change their relative position. From this we may conclude that a large body of water, an ocean, must be part of the problem since the tidal action originates in it. For the purpose of this paper, the flows in the large body of water in which the tidal action originates are not

*Prof. of Hydr. Eng., Univ. of California, Berkeley, Calif.
**Associate Research Engr., Inst. of Eng. Research, Univ. of California, Berkeley, Calif.

included in the term "tidal flow." The problem to be studied is the flow condition in smaller bodies of water which are connected with an ocean and influenced by the water surface variations in this ocean. They will be assumed to be sufficiently small that any direct tidal attraction of the sun and the moon on its water can be neglected as insignificant.

It will thus not be the object of this report to discuss the methods of predicting the tidal cycles of the water surface at any point of the oceans; these cycles will rather be assumed to be known and will be introduced as boundary conditions of the flow in the estuaries, bays and canals under consideration. We will assume, furthermore, that the estuary under consideration is sufficiently small that its influence on the cycle of tidal water surface elevations in the ocean is negligible.

In contrast to other movements of the ocean water, such as those produced by wind, the tides are periodic and can be predicted well in advance for a given location on the basis of past observations. The fluid motion is mainly horizontal and hence can be shown to be the same for all particles on a vertical line. The tidal elevation at a given location can be analyzed into a series whose periods are those of the disturbing forces and are therefore known a priori. The amplitudes and phases of the component tides are determined by harmonic analysis of actual tidal observations over sufficiently long periods.

The larger part of the tidal variation is due to the motion of the moon because of its proximity to the earth. The principal effects are due to variations in the phase of the moon, distance and declination. The relative magnitude of these effects varies in different parts of the world. For the precise prediction of the tide at any place one may need from 20 to 30 component tides. The five principal components are as follows:

M_2 = lunar semi-diurnal with a period of 12.42 hours
S_2 = solar semi-diurnal with a period of 12 hours
N_2 = larger lunar elliptic semi-diurnal with a period of 12.66 hours
K_1 = luni-solar diurnal with a period of 23.93 hours
0_1 = lunar diurnal with a period of 25.82 hours.

One finds that the semi-diurnal tides prevail on the shores of the Atlantic Ocean, the diurnal tide on the coast of China, Alaska and the Gulf of Mexico, and mixed tides on the Pacific Coast of North America.

The tide at any point in the oceans may be represented in the form

$$\eta = \eta_0 + \eta_1 \cos(\sigma t - k_1) + \eta_2 \cos(2\sigma t - k_2) + \ldots$$

where σ is the frequency of the diurnal tide = 0.702×10^{-4} sec^{-1}. On the coast of the Netherlands one finds, for example, that the amplitude of the M_2 tide is approximately 80 per cent of that of the total tide. Under such conditions tidal flow in canals can be approximated by a single harmonic component with a period of about 12 hours. This type of approximation used so successfully and extensively by the Dutch may be quite inadequate when the tide is of the mixed type such as at San Francisco.(1)

While the introduction of this variable water-surface boundary condition is the characteristic outside influence on tidal flows, it is by no means the only one. In the case of canals, for instance, it is very common that such a boundary condition exists not only at one end but at both ends and that the tidal cycles at the two ends are different in shape, phase and amplitude. If a system of canals is under consideration there may be even more openings to the open sea, at which the tidal cycle of the ocean provides boundary conditions for the tidal flow in the canal system.

In many tidal flow problems other types of boundary conditions must be satisfied. It is rather common to many estuaries that a river enters the estuary at the land side. The transition from river to estuary is often so gradual that it becomes impossible to decide where one begins and the other ends. The safest way of handling that problem is then to assume the boundary to be in the river and to introduce the river flow as boundary condition. This condition represents thus a discharge, and not a stage as that at the ocean side. The same condition must be introduced at a dead end of the estuary—the discharge as the flow velocity at that point is zero.

Other boundary conditions may be introduced in other cases. For instance a very long canal may be approximated by an infinitely long canal, which is mathematically much easier to solve than the canal of finite length.

Our tidal flow problem is now defined as the flow of water in a basin which is small compared to the ocean, but directly connected with it, and of which we know some boundary conditions. This flow wants to be described by the water surface elevation and the flow velocity at various points of the system as a function of time. Such a description permits us to predict when and where dangerous or undesirable stages or velocities will occur in the system.

Most canals and estuaries have a limited water depth and the flows in them will incur appreciable friction. In the calculation of the flow the frictional effects can for that reason usually not be neglected. It will be seen that it is the turbulent friction which complicates the mathematical description of tidal flows more than any other single factor.

If a river enters the estuary there will be both sweet river water and salty sea water in the estuary. Density currents, i.e. the relative motion in horizontal layers of water of different density may occur in this case. However, there is an interesting interrelation between turbulent friction and the occurrence of such density currents which will permit us to separate these two effects. Turbulent friction, such as the friction described by Manning's equation or similar others, is characterized by the existence of turbulence in the entire cross section. This turbulence has vertical components which exchange fluid vertically at such a rate that the transfer of momentum between different layers holds the entire shear stresses in equilibrium. Such a turbulent exchange of fluid would naturally mix water layers of different density in a rather short time and make density currents impossible. We can conclude from this that density currents are probably not an important factor where friction is important. In those cases where the turbulent mixing is insufficient to break up density currents, the flow patterns will be so different from that usually described by the common friction laws that these friction laws do not apply. A combination of density currents and frictional effects is thus at the existing stage of our knowledge impossible.

It is a very common occurrence that sediment is moved to the tidal flow basin either by a river or from the open sea. Some of the most important engineering problems in such basins are actually caused by sediment deposition. The question thus arises whether or not the sediment motion as such has an influence on the resulting tidal flows.

For a general appraisal of this effect it is necessary to divide the sediment materials into a fine part, which stays in suspension until it is finally deposited in some dead-water area where it remains permanently, and into a coarser part which moves essentially along the bottom whenever the flow is locally sufficiently strong, and which represents a shifting deposit of loose material. It is clear that both types of sediment, if continuously supplied to the basin, will gradually fill it. This effect is sufficiently slow and gradual,

however, that it has no effect on the flow at a given instance, but that it will only change slowly the geometry of the basin. The main instantaneous influence on the flow comes due to the fact that the bottom is covered by such sediment material and it is thus the sediment material which defines the roughness of the bottom. The fine material in the low flow areas may be observed to deposit as a very smooth and regular surface which results in a very low friction factor. Such areas will afford the flow a minimum resistance unless friction is caused by any foreign matter such as a protruding rock, structures or debris deposited at extremely large flows. The shifting coarser materials on the other hand are rougher in themselves. In addition they often have the tendency to form ripples and bars due to flow and wave action, such that they provide not only for a rougher, but also for a more irregular surface—both factors increasing the effective roughness of the bottom. A careful choice will thus be necessary of friction factors taking into account the distribution of various sediment deposits. In most basins the effective value of such friction factors is actually possible only by direct measurement and by subsequent analysis of some flows.

An additional and very important influence on the flow in tidal basins is wind. Most catastrophic flood conditions in tidal basins that have resulted in levee breaks and other flood damage, were caused at least in part by wind. If a strong permanent wind flows over a water surface it creates a shear force which drives the water in the direction of the wind and makes it assume a steady slope towards the wind to counteract the shear force. Flow velocities will result as a consequence of changes of this wind force. Both the strength and direction of the wind can not be predicted with sufficient accuracy to introduce it as a cause of tidal calculations. Usually it is introduced only as a possible additional effect and taken care of by an additional stage and slope of the water surface.

Other effects, such as temperature differences, which have an effect on the density of the water, may be assumed to be unimportant for the same reason as salinity differences as long as turbulent friction prevails. Animal and vegetable growth, on the other hand, may temporarily or permanently influence both the friction factors and the location of sediment deposits and should be studied in important problems.

The Differential Equations of Tidal Flow

After defining a tidal flow in the preceding paragraphs and after discussing there the various effects and causes which must go into a mathematical description of such flows, the actual equations can be established which describe the flow. Let us first see if there are any other problems which are solved and may lead the way for a general solution of the tidal flow problem.

The "tidal flow" was defined as the flow in a basin, canal or canal system, which is at least at one point open to an ocean with known tidal action, which may or may not have other openings with steady or fluctuating discharges and which is often sufficiently large for its depth to develop significant friction. At the opening to the ocean the water surface elevation is varying in a periodic fashion due to tidal action. This will cause a wave action in the tidal basin originating from that point suggesting that the problem be treated as a wave problem. At the same time flows will occur between the ocean and the basin which will constantly change magnitude and direction and these flows will propagate into the basin—suggesting that the problem may be interpreted as one of flood routing, especially since the period of the tide is very long

compared with most other periodic waves. Fortunately the differential equations describing these two problems are identical such that we are free to interpret the flow as a wave problem or as an unsteady flow problem.

Before the differential equations of this unsteady or wave motion can be established it is necessary to classify the basin under consideration in a general manner. This classification is easiest understood if the problem is interpreted as a wave problem. Everybody has seen the wave pattern which is generated by a stone thrown into a still pond. Circular wave fronts will emerge from the point of origin and spread evenly over the entire water surface in all horizontal directions. Let us call such a pattern three-dimensional since the wave, which is a displacement of the water surface in the vertical direction, spreads in both horizontal directions no matter how the horizontal axes are chosen.

The two-dimensional wave motion on the other hand is one in which most of the wave propagation occurs in one preferred direction with all wave fronts normal to this direction. Such conditions are found predominantly in canals which are long compared to their width and where the source of the wave motion is located at the end of the canal. Due to reflection at the side walls such waves will have the tendency to orient themselves such that the crests become straight and normal to the channel axis. This statement is only partially correct for short waves, i.e. for wave lengths short with respect to the channel widths, but becomes more and more generally correct as the wave length becomes large with respect to the channel width. This statement calls for an estimate of the tidal wave lengths, for the purpose of comparison with the widths. The wave length may be derived as the product of wave period (something over 12 hours) and the wave velocity which for this purpose may be estimated by the shallow water wave propagation formula for frictionless flow, $c = \sqrt{gy}$, where y is the still water depth and g the acceleration due to gravity. The wave lengths resulting for depths of 3, 10 and 50 feet are 85, 150 and 340 miles, respectively. It will be agreed that these wave lengths are long with respect to the widths of most tidal basins; it will become rather interesting in the application of any resulting formulas that in most practical cases the wave length is of equal order of magnitude or even longer than the length of the tidal basin in which it propagates. The order of magnitude of wave period and wave length are the same for flash floods in medium sized streams and the resulting flows must be expected to cover also similar ranges of conditions. As all such flood flows are calculated by means of the hydraulic approach, it can be expected that the same approach, which essentially interprets the flow as two-dimensional, is also sufficient for many tidal problems. Up to this date only two-dimensional problems or those borderline cases which are three-dimensional but can be approximated with sufficient accuracy by a two-dimensional description have been studied analytically in the U.S.

In this section we thus derive equations for the velocity and surface elevation of water in a channel as functions of position and time. These equations are not specifically concerned with tidal flow. The influence of the tide enters the solution of these equations through the boundary conditions imposed at the ocean entrances. It is clear that by introducing other boundary conditions one could in a similar manner solve problems such as flood routing. In fact some of the techniques of solution are applicable to quite general boundary conditions. It is only due to the approximate periodicity and the relatively small flow velocities of many tides that appreciable simplifications are possible.

In order to study the details of varied flow such as velocity distributions, one generally uses a hydrodynamic approach in which local values are solved for. In many cases a different approach is used either because specific details are not important or because hydrodynamic solutions are not feasible. This latter approach we call hydraulic. The entire cross section is considered as a stream tube of variable dimensions and the flow in it is described by mean values over a cross section only. One must be careful in carrying out the averaging, however. The best way of doing this is to proceed directly from the hydrodynamic equations and to integrate over a cross section. For such a derivation we refer the reader to a paper by Keulegan and Patterson.[2]

We adopt a coordinate system in which the x-axis is horizontal, generally following the center line of the channel bed. The y and z axes are perpendicular to x and lie in the plane of the channel cross section, y being horizontal and z vertical.

We shall assume, as in[2] that:

1) The fluid is homogeneous. We specifically neglect thereby density currents, sediment movement, etc.
2) The fluid is incompressible.
3) The channel slope $i = \tan\theta$ is small so that $\tan\theta \simeq \theta \simeq \sin\theta$.
4) The flow is gradually varied. This means that the second derivatives of the cross-section A and of the mean velocity U and the squares and products of their first derivatives may be neglected in comparison with the first derivatives. In general this requires that the banks of the channel be not much flatter than a 1:1 slope, that the slope of the wave profile $\ll 1$ and that the curvature of the wave profile $\ll \frac{1}{u}\frac{\partial u}{\partial x}, \frac{1}{u^2}\frac{\partial u}{\partial t}$ where u is the mean velocity over a cross section. In such cases we can neglect the components of acceleration parallel to a cross section.

Let U (x,y,z,t) be the x-component of the local flow velocity and A (x,t) be the cross sectional area. We denote by

$$Q = \int_A U dA = \text{discharge}$$

$$u = Q/A \quad = \text{mean velocity.}$$

Consider now the conservation of fluid for a small volume $\Delta V = A(x,t)\Delta x$ of cross sectional area A(x,t) and thickness Δx. (Fig. 1)

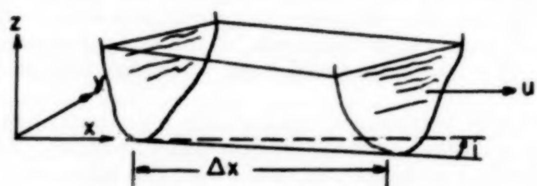

Fig. 1: Channel Section.

The net outflow of water in ΔV per unit time

$$= Q(x + \Delta x) - Q(x) \approx + \frac{\partial Q}{\partial x}\Delta x$$

The increase in volume per unit time

$$= \frac{\partial A}{\partial t} \Delta x .$$

Since the fluid is incompressible, net outflow plus increase in volume equals zero. Thus

$$\frac{\partial (uA)}{\partial x} + \frac{\partial A}{\partial t} = 0 . \tag{1}$$

This equation applies to canals or estuaries alike if lateral flows are neglected. If, however, the local velocity changes very much over a cross section such as in an estuary, the quantity u loses its power as a parameter to describe the flow sufficiently. Of particular interest in such a connection is an estuary containing a narrow, deep channel and adjacent extensive shoals which are flooded when the tide rises. Another such case of interest is a river containing groins, the water circulating in eddies in the region between the groins in such a way as to guide the flow in the stream bed. In such cases it is convenient to divide the total cross section into a stream bed in which the flow velocity is relatively high and a lateral basin which contributes only to storage and not to discharge. In general the water velocities and water levels are quite different in the stream bed and the lateral basins. If we denote by A the area of the stream bed and A_B the area of the lateral basin, the general continuity equation is (see Fig. 2).

$$\frac{\partial (uA)}{\partial x} + \frac{\partial A}{\partial t} + \frac{\partial A_B}{\partial t} = 0 .$$

Assume that we are dealing with a situation such as that pictured below:

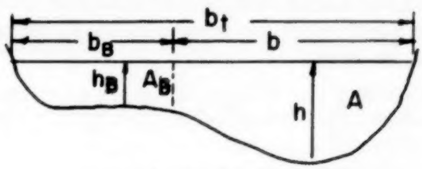

Fig. 2: Cross Section

If we assume that the water level has time to adjust laterally, then

$$\frac{\partial A_B}{\partial t} = b_B \frac{\partial h_B}{\partial t} + h_B \frac{\partial b_B}{\partial t} = b_B \frac{\partial h}{\partial t} + h_B \frac{\partial b_B}{\partial t}$$

where the average width is $b_B = \frac{A_B}{h_B}$. Assume that b_b is a constant so that the lateral basin has a rectangular cross section. Then

$$\frac{\partial A_B}{\partial t} = b_B \frac{\partial h}{\partial t}$$

Thus the continuity equation becomes

$$\frac{\partial (uA)}{\partial x} + b_t \frac{\partial h}{\partial t} + h \frac{\partial b}{\partial t} = 0$$

where $b_t = b + b_B$. If moreover the stream bed has a rectangular cross section

$$\frac{\partial(uA)}{\partial x} + b_t \frac{\partial h}{\partial t} = 0 \qquad (2)$$

This is essentially the same as (1) except for the lateral discharge term and where instead of the width of the stream bed we use the total width b_t.

Consider now the following contributions to the x-component of momentum per unit time for the volume ΔV:

1) storage of momentum = $\Delta x \, \rho \frac{\partial}{\partial t} \int_A U \, dA = \rho \Delta x \frac{\partial(uA)}{\partial t}$

2) net outflow of momentum = $\rho \int_{A+\Delta A} U^2 dA - \rho \int_A U^2 dA \simeq \rho \Delta x \frac{\partial}{\partial x}(\alpha A u^2)$

where $\alpha = \frac{\int_A U^2 dA}{u^2 A} \geq 1$

3) force due to pressure gradient = $-\Delta x \int_A \frac{\partial p}{\partial x} dA = -\Delta x A \frac{\partial p}{\partial x}$

4) resistance force due to shear on wetted perimeter = $-\Delta x \int_x \tau dx$

where τ = shearing force on wetted perimeter x.

This can be written in the form $\Delta x \frac{\rho u^2 x}{C^2}$

where C is defined by $\tau_0 = \frac{\int_x \tau dx}{x} = \frac{\rho u^2}{C^2}$.

We assume that the shear on the free surface is zero.

We neglect any additional terms arising for estuaries with storage basins. For small velocities such terms are generally small enough to be neglected.

Collecting results we have

$$\frac{\partial}{\partial t}(uA) + \frac{\partial}{\partial x}(\alpha A u^2) = -\frac{A}{\rho}\frac{\partial p}{\partial x} - \frac{g u^2 A}{R C^2} \qquad (3)$$

where R = A/x hydraulic radius.

Neglecting the components of acceleration parallel to a cross section we find that the pressure p is given by the hydrostatic formula

$$p = \rho g (H - z) \qquad (4)$$

where H = surface elevation. This formula is valid for waves whose wave length is large compared to the depth of water (3). Substituting (4) into (3) we get

$$\frac{\partial}{\partial t}(uA) + \frac{\partial}{\partial x}(\alpha A u^2) = -Ag\frac{\partial H}{\partial x} - \frac{g u^2 A}{C^2 R} \qquad (5)$$

Multiplying the continuity equation (1) by αu and subtracting from the above we find

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + (1-\alpha)\frac{u}{A}\frac{\partial A}{\partial t} = -g\frac{\partial H}{\partial x} - \frac{g u^2}{C^2 R}. \qquad (6)$$

For $\alpha = 1$ this reduces to

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial H}{\partial x} - \frac{gu|u|}{C^2 R} \quad (7)$$

The coefficient, α, appears to depend to a large extent on the width depth ratio. The numerical values are only slightly larger than one in most cases (see reference 4, p. 272). We have written $u|u|$ in place of u^2 since for reversing flows the frictional resistance must be proportional to u^2 and have the same sign as u.

The hydraulic radius, R, is a function of x and t. To illustrate the variation of R consider a canal of trapezoidal cross section as shown in Fig. 3.

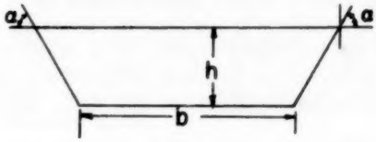

Fig. 3: Trapezoidal Section.

Then

$$R = \frac{A}{X} = h\,\frac{1+(h/b)\,\mathrm{COT}\,\alpha}{1+(2h/b)\,\mathrm{CSC}\,\alpha}$$

For an example we find for the Panama sea-level canal[5] that for $\tan\alpha = 1$, b = 600 feet, h = 60 feet, the hydraulic radius is R = 0.86 h. Moreover, for a tidal range of 20 feet and water depths of 40, 50 and 60 feet, R is 36, 44 and 52 feet respectively.

Solution of the Differential Equations

The differential equations (1) or (2) and (6) or (7) include most tidal and flood flows which can be described as hydraulic problems. There is no limitation on the shape of the channel other than its adaptability to hydraulic flow description and constant width over the range of tidal stages. There is no restriction on the shape of the tide or on the hydrographs which may create the flows and stages under consideration. Equations (1) or (2) and (6) or (7) are thus so general and all-inclusive that their knowledge is in itself not a great help. Only a solution of these equations which satisfies the boundary conditions of the particular channel and of the given tide cycle and hydrographs, gives actually the water surface lines and flow velocities which the engineer needs for the design of the canal and its works. To find the explicit solution of the equations for a given set of boundary conditions is in the general case impossible in closed form. Most generally applicable, but also most time consuming, are various numerical step-methods. To the writers' knowledge, this approach has not been used in the U.S. to any significant amount in tidal flow calculations.

The other method of approach is the introduction of simplifying assumptions; the most common are:

1) Average elevation of water surface constant along horizontal canal.
2) Simple harmonic tide.
3) Linear friction law.
4) Constant rectangular channel cross-section.

5) Nonlinear terms are neglected in the differential equations.

Various solutions are possible on this basis. Chronologically, the first was made by W.B. Parsons on the occasion of the solution of tidal flow problems for the Cape Cod Canal.[6]

Parsons' Harmonic Theory

This is based on the introduction of all five approximations enumerated above. In this case an exact solution of the equations (1) and (7) is possible and was given for the case of two different tidal cycles applied to the two ends of a uniform horizontal canal. While approximations 1, 2, and 4 are easily understood, it may be necessary to explain the meaning of approximations 3 and 5. It is clear that the turbulent friction in a canal will always be described by an energy slope which is about proportional to the square of the average velocity u. For a particular flow one may, however, approximate the term

$$\frac{g\,|u|\,u}{C^2 R} \quad \text{by} \quad C'\,u$$

where C' is a constant equal to a representative value of $g\,|u|\,/\,C^2 R$, such that the work done by the linearized friction equals that of the quadratic friction of equation (7) over a full cycle. This changes (7) into

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + g\,\frac{\partial H}{\partial x} + C'u = 0 \tag{8}$$

while equation (1) may be written as

$$\frac{\partial}{\partial x}\left[uH\right] + \frac{\partial H}{\partial t} = 0 \tag{9}$$

since the width b was assumed to be constant over the range of tidal stages. For sinusoidal tides the principle of equivalent work gives

$$C' = \frac{8}{3\pi}\,g\,|u_{max.}|\,/\,C^2 R$$

The factor C' is in general a function of x, but is often averaged over a given reach for convenience of calculation.

Both equations are non-linear in u, H and their derivatives. It can be shown, however, that these equations can be linearized for tidal flow problems if the small terms are eliminated. Such an appraisal can be based on the fact that all the waves under consideration are smooth and very long for their height. Let us introduce an auxiliary small constant parameter ϵ which has the order of magnitude of the ratio of the wave height divided by the wave length of the tidal wave.

Now let us develop the variables as follows

$$\begin{aligned} u &= u_0 + \epsilon u_1 + \epsilon^2 u_2 + \ldots\ldots \\ H &= H_0 + \epsilon H_1 + \epsilon^2 H_2 + \ldots\ldots \end{aligned} \tag{10}$$

Such u_0 and H_0, respectively, are the values of u and H for zero wave steepness. Introducing these expressions into (9), we find for equation (9)

$$u_0\frac{\partial H_0}{\partial x}+H_0\frac{\partial u_0}{\partial x}+\frac{\partial H_0}{\partial t}+ \epsilon\left[u_0\frac{\partial H_1}{\partial x}+u_1\frac{\partial H_0}{\partial x}+H_0\frac{\partial u_1}{\partial x}+H_1\frac{\partial u_0}{\partial x}+\frac{\partial H_1}{\partial t}\right]+\epsilon^2\cdots=0 \tag{11}$$

This equation (11) must be fulfilled also for the case of zero wave, which is represented by $\epsilon = 0$, and can thus be divided into two independent equations

$$\begin{aligned} &u_0\frac{\partial H_0}{\partial x}+H_0\frac{\partial u_0}{\partial x}+\frac{\partial H_0}{\partial t}=0 \\ &u_0\frac{\partial H_1}{\partial x}+u_1\frac{\partial H_0}{\partial x}+H_0\frac{\partial u_1}{\partial x}+H_1\frac{\partial u_0}{\partial x}+\frac{\partial H_1}{\partial t}=0 \end{aligned} \tag{12}$$

In order to obtain Parsons' equations we assume that the average velocity u_0 in these linear equations between u_1, H_1 and their derivatives vanishes.

With $u_0 = 0$, the first equation (12) gives $\partial H_0/\partial t = 0$, which is correct anyway, since H_0 is an average value.

The first and the fourth term of the second equation disappear too, leaving of equation (12)

$$u_1\frac{\partial H_0}{\partial x}+H_0\frac{\partial u_1}{\partial x}+\frac{\partial H_1}{\partial t}=0 \tag{13}$$

Equation (8) may also be linearized by the use of (10) and gives

$$\begin{aligned} &\frac{\partial u_0}{\partial t}+u_0\frac{\partial u_0}{\partial x}+g\frac{\partial H_0}{\partial x}+C'u_0=0 \\ &\frac{\partial u_1}{\partial t}+u_0\frac{\partial u_1}{\partial x}+u_1\frac{\partial u_0}{\partial x}+g\frac{\partial H_1}{\partial x}+C'u_1=0 \end{aligned} \tag{14}$$

In the first equation (14) all terms except the third vanish because of $u_0 = 0$ - from which $\partial H_0/\partial x = 0$. In the second equation the terms with u_0 disappear, leaving altogether from (13) and (14)

$$\begin{aligned} &H_0\frac{\partial u}{\partial x}+\frac{\partial H}{\partial t}=0 \\ &\frac{\partial u}{\partial t}+g\frac{\partial H}{\partial x}+C'u=0 \end{aligned} \tag{15}$$

if the indices | are dropped as insignificant.

H_0 is herein the constant undisturbed water depth.

H is the deviation of the actual water surface from the zero level.

u is the instantaneous water velocity.

Equations (15) can be solved in closed form as Parsons showed. The solution is rather easily obtained because of the linear character of equations (15) which permits the construction of the general solution by super-position of particular solutions. Parsons shows that the general solution for tidal flows in a horizontal canal with sinusoidal tides of equal frequency, but different phase and amplitude acting at the two ends may be given as the sum of four progressive damped waves. For the mathematical expressions, the reader may be referred to Parsons' paper, since a duplication of the long expressions would serve no useful purpose at this place.

Parsons applied his solution to the Cape Cod Canal and, after judicious choice of an appropriate friction factor C', was able to check rather closely the measured water profiles along the canal for various phases of the tidal

motion. The proper choice of the friction factor is most important if a reliable result is desired. A few remarks are thus in place about the choice of C'. The "linearized" friction factor C' is actually not a constant for any canal, but is changing in reality with the instantaneous flow velocity. If equation (15) applies an average of the instantaneous C' values, some of the details of the flow pattern may be lost, but its general shape is not much affected by this approach.

The major advantage of equations (15) is their linear character which permits the superposition of individual solutions. Here arises the question of what velocity must be used to determine C' for the individual solutions. There is the tendency of applying for each individual partial wave its own average velocity for the determination of its C' value (see Brown's reflected wave theory, which is discussed next). But that is wrong. A reasonably correct result can be obtained only if the resultant velocity of all component waves is used to derive C'. This can most easily be seen if a wave is thought to be composed of two identical waves with equal phase and period as the original wave, but of half the amplitude. All other terms being truly linear, add from the two partial waves to exactly the same amount as determined for the original wave, all except the friction term in which both the velocity and C', which itself contains u, are reduced to half the original value. The friction term reduces thus to 1/4 of the original value, and the sum of the two waves still has a friction term which is only half of the original value—proving this method to be wrong. It is easily seen, on the other hand, that the full friction term is obtained if the resultant velocity is used in the determination of C'. This way the inherently non-linear character of the friction term is preserved. This way there exists a mutual interference between the individual partial solutions. The proper C' value can in most cases be obtained only by trial and error.

Fig. 4 gives for example the solution of equations (15) to the model of a sea-level Panama Canal as described by Meyers and Schultz. It may be seen there that for instance the flow velocity at one end may be calculated with good accuracy from the vertical tide patterns at the two ends and a properly chosen C -value.

Several such calculations were made in the past and instantaneous water surfaces were determined. For all canals which were short compared to the wave length as estimated previously, the profiles with high surface slope turned out to be practically rectilinear. This fact gives Pillsbury a valuable starting point in his calculation method.

In conclusion, it may be said that Parsons' method is correct and reliable where it applies, i.e. for sinusoidal tide applied to uniform horizontal canals without average flow in one direction. Whenever any of these conditions are significantly violated, one must change the form of the differential equations—and naturally of the solutions.

Brown's Reflected Wave Theory

Tidal flow problems which are described by linear differential equations can be solved by superimposing the motion of simple progressive waves, and introducing the effect of abrupt changes in cross section. This is in essence the method proposed by Brown.[7] However, the actual solutions proposed by Brown appear unnecessarily complex and at the same time theoretically oversimplified.

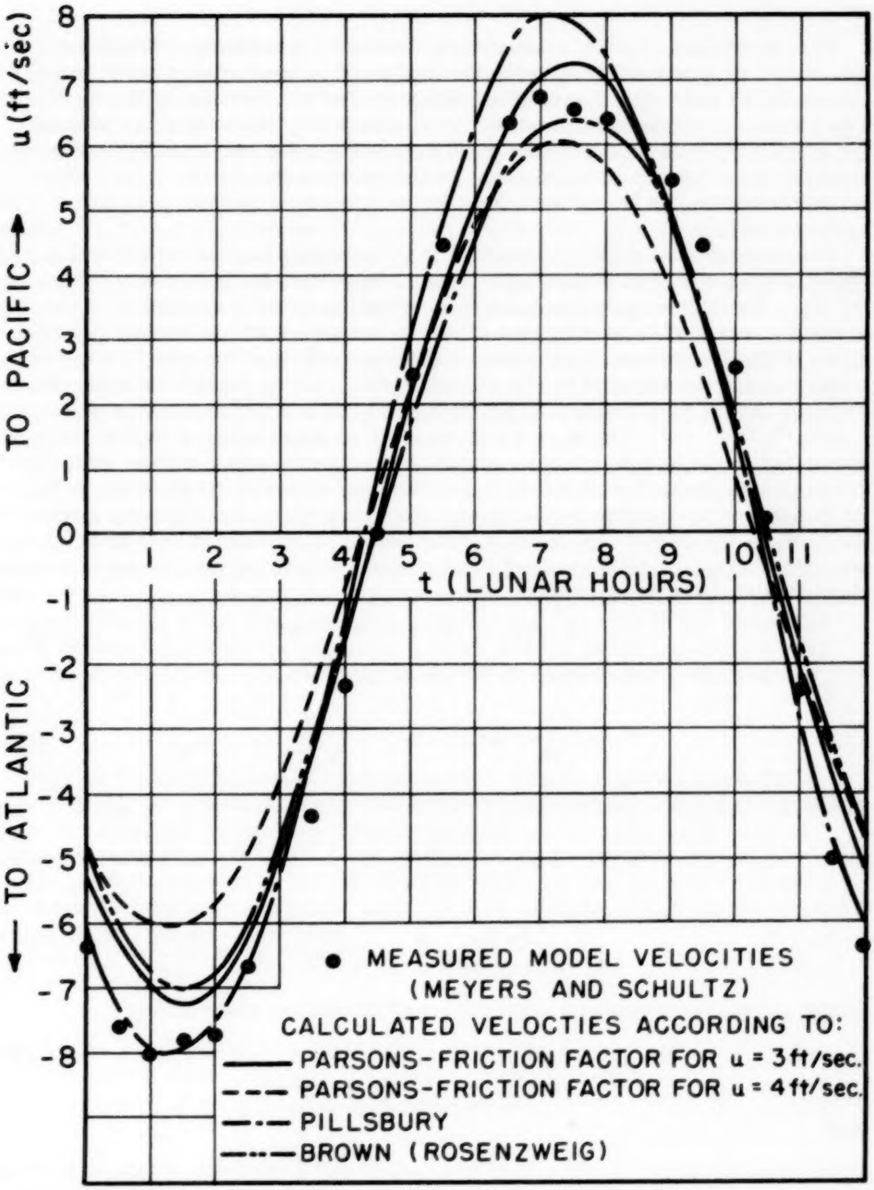

Fig. 4

Brown assumes that the effect of friction is small so that it can be considered as a small change referred to a sinusoidal frictionless wave. The frictional resistance is taken as proportional to the product of particle velocity and wetted perimeter. Equating the work done by the frictional resistance to the loss of energy by a sinusoidal wave one finds, on integrating the resulting equation, a series approximation for the change of wave height with travel distance.

This derivation appears unnecessary, however, for assuming frictional resistance proportional to velocity the motion of a simple progressive wave can easily be determined exactly as demonstrated for instance by Parsons. The phase of the tide is there shown to be affected by the friction as well as the amplitude, thus contradicting Brown's assumptions. Moreover, the flow velocity is not simply given in terms of the vertical tide by the formula for frictionless flow, as Brown supposes, but is given by more complicated expressions.

Consider now the partial reflection of an incident progressive wave at a place of abrupt change in section of a canal from breadth and depth b_1, H_1 to b_2, H_2. The following determination of reflection at these changes of cross section is reasonable at distances from the change which are moderate multiples of the transverse dimensions of the canal. Within this region we assume that the net effect of friction is negligible. Under these conditions the solution for the amplitudes of the transmitted and reflected waves is well known.[3, Sect. 176] The analysis given in [3] is essentially as follows. Assume that at the change in cross-section mass is conserved and the wave surface is continuous. Denote by a_i, a_r, a_t the amplitudes of the incident, reflected and transmitted waves respectively. For an incident wave moving from depth H_1 to H_2 we have the well-known equations for a frictionless progressive tidal wave in a uniform horizontal channel with H the instantaneous local depth and H_o the average depth.

$$H - H_o = a \sin(kx - wt) \tag{16}$$

and the corresponding velocity of the water particles

$$u = a\sqrt{\frac{g}{H_o}} \sin(k_x - wt) = \sqrt{\frac{g}{H_o}} (H - H_o) \tag{17}$$

and at x = 0 depth change actually occurs

$$\sqrt{\frac{g}{h_1}} b_1 h_1 (a_i - a_r) = \sqrt{\frac{g}{h_2}} b_2 h_2 , \quad a_i + a_r = a_T .$$

Thus

$$T - R = 1 , \qquad \frac{b_2 h_2}{C_2} T + \frac{b_1 h_1}{C_1} R = \frac{b_1 h_1}{C_1}$$

where the transmission and reflection coefficients are respectively

$$T = \frac{a_T}{a_i} , \qquad R = \frac{a_r}{a_i} \tag{18}$$

and

$$C_1 = \sqrt{gh_1} , \qquad C_2 = \sqrt{gh_2}$$

These equations have the solution

$$R = \frac{b_1 C_1 - b_2 C_2}{b_1 C_1 + b_2 C_2} , \quad T = \frac{2 b_1 C_1}{b_1 C_1 + b_2 C_2} . \tag{19}$$

One easily verifies that the sum of energies in the reflected and transmitted waves is equal to the energy in the incident wave, i.e. $b_1C_1R^2 + b_2C_2T^2 = 1$. Although we have used periodic waves to compute R and T this is not essential in the derivation (see ref. 3).

As particular cases we see that when $b_2 \to 0$, $R \to +1$ and when $b_2 \to \infty$, $R \to -1$. Thus at a closed end a progressive wave reflects with the same sign and equal amplitude. At an open end it reflects with opposite sign and equal amplitude.

Thus for a canal which consists of sections of constant cross section we have a complete scheme for dealing with linear wave motions. Within each individual section the roughness may vary. Starting with conditions at one end we proceed along the canal in small increments, advancing the solution step by step by Parsons' equations for progressive waves. At a change in cross section, including an open or closed end we use formulae (19) to determine the new amplitudes and assume periods and phases are unaffected by the transition. The resultant tides are simply formed by algebraic addition of successively reflected and transmitted waves due to tides acting at the two ends of the canal. If the canal has essentially constant characteristics explicit solutions are available as in Parsons. One should note that the friction coefficient must be estimated from the total velocity and not for each individual wave, as Brown proposes.

Brown's method is thus based on the very promising idea of treating the tidal flows as a system of progressive waves. The method has been applied to various existing canals and gave good results for the shape of the water surface profiles. It must be repeated that in most channels which are all short compared with the wave length, the profiles are usually straight lines. With the water surface elevations given at both ends, the profile is bound to be correct. Rather large deviations from measurement result in the velocities which depend heavily on the proper description of the frictional effects. But there is no reason why these shortcomings could not easily be eliminated.

Pillsbury's Theory

Since previously described methods appeared to be unsatisfactory, Pillsbury[8] proposed a method of successive approximations for dealing with tidal flows in canals and estuaries. The corrections are based on the so-called "primary currents."

The primary current is derived by first assuming that the vertical tide, H', has a known amplitude and phase while the flow velocity, u, has the same period as H'. Thus assume that

$$H' = H - H_o = h\cos(\sigma t + \gamma) \tag{20}$$

$$u = B\sin(\sigma t + \beta) \tag{21}$$

where h, γ, σ are known. Then by (20) the slope is of the form

$$\frac{\partial H'}{\partial x} = A\cos(\sigma t + \alpha) \tag{22}$$

where A, α can be computed from H, γ.

Substituting (21), (22) into (20)

$$\sigma B\cos(\sigma t + \beta) + gA\cos(\sigma t + \alpha) + k_o \zeta B\sin(\sigma t + \beta) = 0 \tag{23}$$

Equating coefficients of $\sin(\sigma t)$ and $\cos(\sigma t)$ separately to zero

$$\sigma B\cos\beta + gA\cos\alpha + k_o\zeta B\sin\beta = 0$$
$$-\sigma B\sin\beta - gA\sin\alpha + k_o\zeta B\cos\beta = 0$$

Solving these equations for B and β we find

$$\tan\beta = \frac{-k_o\zeta\cot a-\sigma}{k_o\zeta-\sigma\cot a} \tag{24}$$

$$B = \frac{gA}{k_o\zeta}\ \sin(a-\beta) \tag{25}$$

The variables A, α are computed at the middle of the canal from the entrance tides by assuming that the instantaneous profiles are straight lines. Thus denoting by H_0' and H_1' the entrance tides

$$A\ \cos(\sigma t + a) = \frac{H_1'-H_0'}{L} \tag{26}$$

This assumption is nearly correct for short canals such as the Panama Canal (L~42 miles) as we noted previously. The amplitude and phase of the flow velocity at the middle of the canal are then computed from (24) (25).

Flow velocities at other locations in the canal are obtained by summing known velocity increments. Thus writing the continuity equation in finite difference form

$$\Delta u \simeq -\frac{\Delta x}{H_o}\frac{\partial H'}{\partial t} = \frac{\Delta x}{H_o} h\sigma\ \sin(\sigma t + \gamma) \tag{27}$$

We add these increments of velocity to the value calculated at the center of the canal in order to find the velocity at other locations. Thus this velocity will be of the form

$$u = B\sin(\sigma t+\beta) = B_m\sin(\sigma t + \beta_m) + \sum\frac{\Delta x}{H_o} h\sigma\ \sin(\sigma t+\gamma)\,. \tag{28}$$

Hence

$$\begin{aligned} B\sin\beta &= B_m \sin\beta_m + \sum\frac{\Delta x}{H_o} h\sigma\ \sin\gamma \\ B\cos\beta &= B_m \cos\beta_m + \sum\frac{\Delta x}{H_o} h\sigma\ \cos\gamma\,. \end{aligned} \tag{29}$$

Further adjustments of the elevation differentials are made so that their sum is the difference of the entrance elevations.

On the basis of the primary current computed in the manner described above, Pillsbury derives distortions due to the velocity head term in (7), the minor components of the friction and variations of the hydraulic radius, R, and Chezy coefficient K. Consider the correction i to the approximate velocity B sin(σt +β) due to friction alone, i.e. let

$$u = B\sin(\sigma t + \beta) + i\ . \tag{30}$$

Neglecting the velocity head term in (7) and assuming $\frac{\partial H}{\partial x}$ is given by (22) we find on substituting into (7) that

$$B\sigma\ \cos(\sigma t+\beta) + \frac{\partial i}{\partial t} + gA\cos(\sigma t+a) + \frac{g}{K^2R}\left[B\sin(\sigma t+\beta)+i\right]|B\sin(\sigma t+\beta)+i| = 0\,. \tag{31}$$

Now B, β are chosen so that the first equation of (29) is satisfied. Hence

$$B\sigma\cos(\sigma t+\beta) + gA\cos(\sigma t+a) + k_o\zeta B\sin(\sigma t+\beta) = 0 \tag{32}$$

Thus by (31) and (32)

$$\frac{\partial i}{\partial t} - k_o\ \zeta B\ \sin(\sigma t+\beta) \pm \frac{g}{K^2R}\ (B\sin(\sigma t+\beta)+i)^2 = 0$$

Expanding the last term in this equation, neglecting i^2 and also the effect of i on the sign of the velocity, one obtains a linear differential equation for i which is readily solved. Not realizing the possibility of an explicit solution, Pillsbury resorts to a laborious numerical solution by finite differences. In a similar way additional corrections can be obtained by setting $u = B \sin(\sigma t + \beta) + i + j$ and expanding.

This procedure of Pillsbury's for computing corrections to a primary current independently for velocity, friction head, etc., does not appear correct. Moreover, and this seems to be the largest source of error, the elevation and slope can not be simply assumed to be of the form (20), (22) in computing these corrections. In fact, one of the principal effects of the nonlinear terms in the differential equations (2), (7) is to introduce additional harmonics in both H and u.

Again, the same remarks can be made as for Brown's method. The shortness of most canals prevents any shortcomings of the method from influencing the water surface profiles much. Again, the velocities show the larger deviations, although an approximate choice of friction factors can go far in overcoming any basic difficulties of the method.

Final Remarks

It is remarkable how small the number of actually performed tidal flow calculations in the U.S. is, especially since work has been done on large numbers of problems in this particular field. The writers have not been able to find any such calculations on estuaries, i.e. non-uniform channels. This may be explained, at least in part, by the large size of the projects involved: There existed a certain reluctance to base the large and important projects on approximate and partially oversimplified calculation methods as long as the solution by the "more trustworthy" model could be found. The writers do not want to close this paper without pointing to the main advantage of the calculation: even rather complicated calculations are many times less expensive than a model and can show many results at least as clear and distinct if executed by an experienced man or group.

REFERENCES

1. Marmer, H.A.; Tides and Tidal Currents; Bull. 85, Nat. Research Council, Chapter 7, pp. 229-309 (1932).

2. Keulegan, G.H. and Patterson, G.W.; Effect of Turbulence and Channel Slope on Translation Waves; Journ. Res. Nat. Bur. Stand., 30, R.P. 1544 (1943).

3. Lamb, M.; Hydrodynamics; 6th ed., Dover Publ., N.Y. (1945).

4. O'Brien, M.P. and Hickox, G.H.; Applied Fluid Mechanics; McGraw-Hill Book Co. (1937).

5. Meyers, J.S. and Schultz, E.A.; Tidal Currents; Proc. A.S.C.E. 74, 501 (1948).

6. Parsons, W.B.; The Cape Cod Canal; Trans. A.S.C.E., 82, 138 (1918).

7. Brown, E.I.; Flow of Water in Tidal Canals; Trans. A.S.C.E., 96, 749-834 (1932).

8. Pillsbury, G.B., Tidal Hydraulics; Professional Paper of the Corps of Engineers, No. 34, U.S. Govt. Printing Office (1940).